MATRIX PRODUCT STATES

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INTRODUCTION

Quantum physics reveals a fundamental and bizarre reality, introducing many challenges in understanding the universe. In this project we ask:

How can we efficiently capture the entanglement information of a ground state?

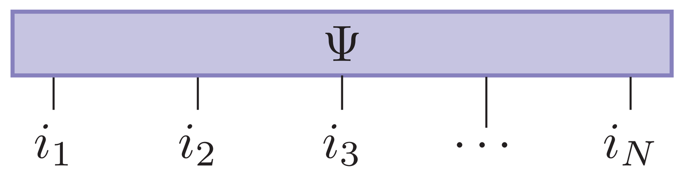
In particular, we investigate the **matrix product state** (MPS) as a computationally efficient method of representing ground states of one-dimensional quantum spin-chains.
Interestingly, MPS do not always exhibit properties associated with **quantum phase transitions** (QPT).

MATRIX PRODUCT STATES

Individual sites in a superposition of d possible states are modelled by a complex Hilbert space $\mathbb{C}^d$. Tensor products model the Hilbert space of the whole system. The state of the whole system is therefore:

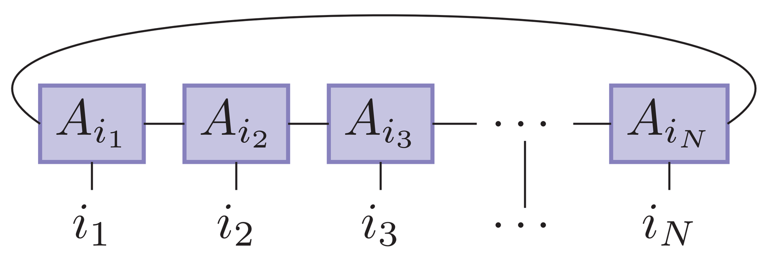
$$|\Psi\rangle = \sum_{i_1 \dots i_N} \psi_{i_1 \dots i_N} |i_1 \dots i_N\rangle$$

Which has a visual representation as a tensor with N indices representing each site.

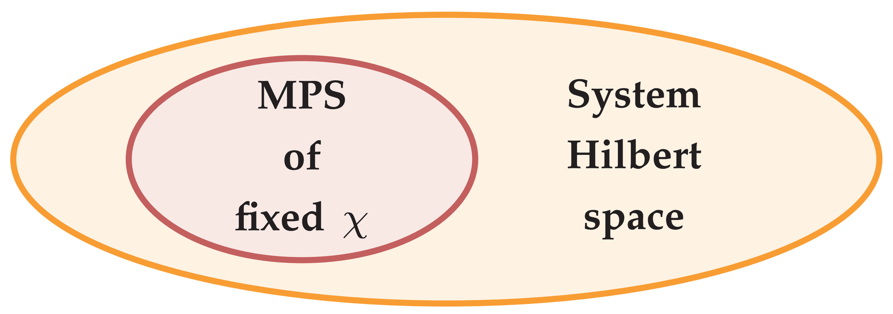


A MPS has the form and visual representation:

$$|\Psi\rangle = \sum_{i_1 \dots i_N} \mathrm{Tr}[A_{i_1} \dots A_{i_N}] |i_1 \dots i_N\rangle$$



Where the A_i are $\chi \times \chi$ matrices. The size of χ is a measure of the entanglement present in the MPS. In general, MPS with finite χ approximate a system by limiting entanglement. For finite χ, MPS form a subspace of the total Hilbert space:



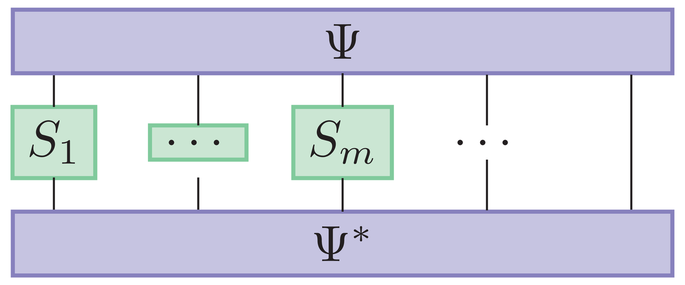
MPS CORRELATIONS

A key element of studying interesting quantum-mechanical ground states are correlation functions. Consider m observables S_i acting on part of the spin chain.

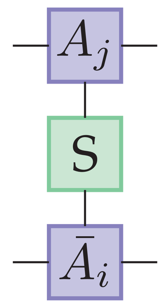
In Dirac notation the correlation function is given by:

$$\langle S_1 \dots S_m \rangle = \langle \psi | S_1 \dots S_m | \psi \rangle \qquad (\dagger)$$

This has a visual depiction:



For MPS we can define a **transfer operator** $E_S = \sum_{i,j} \langle i | S | j \rangle A_j \otimes \bar{A}_i$ which can be thought of as a 4-index tensor that takes 'vertical slices' of the correlation function. The pictorial form of a transfer operator is shown on the right.



By using transfer operators, correlation functions for MPS can be calculated efficiently using the formula:

$$\langle S_1 \dots S_m \rangle = \mathrm{Tr}[E_I^{N-m} E_{S_1} \dots E_{S_m}]$$

This is equivalent to ($\dagger$).

The matrix power $E_I{}^{N-m}$ in the expression can be calculated by diagonalising the matrix and using the eigenvalues of E_I. The resulting expression will therefore be a rational function in terms of eigenvalues of E_I.

Two-point correlations may be considered by setting the observables $S_2 = \dots = S_{m-1} = I$.

EIGENVALUE AND CORRELATION GRAPHS

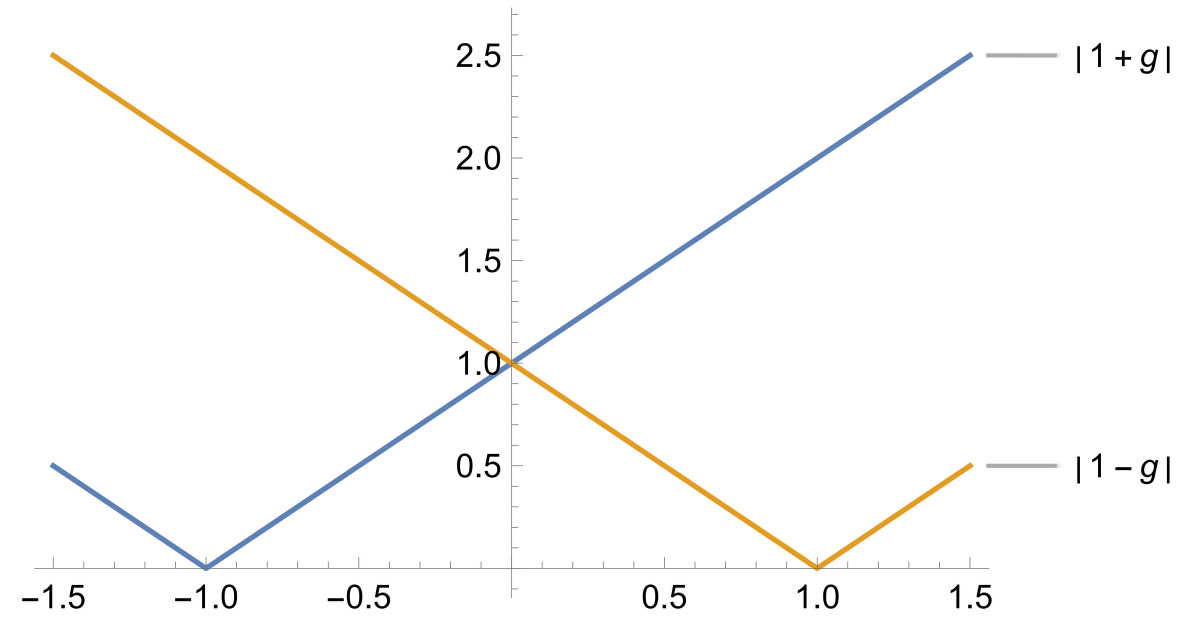


Figure 1: Graph of eigenvalues of E_I for an example from [1], the largest eigenvalue changes when $g = 0$

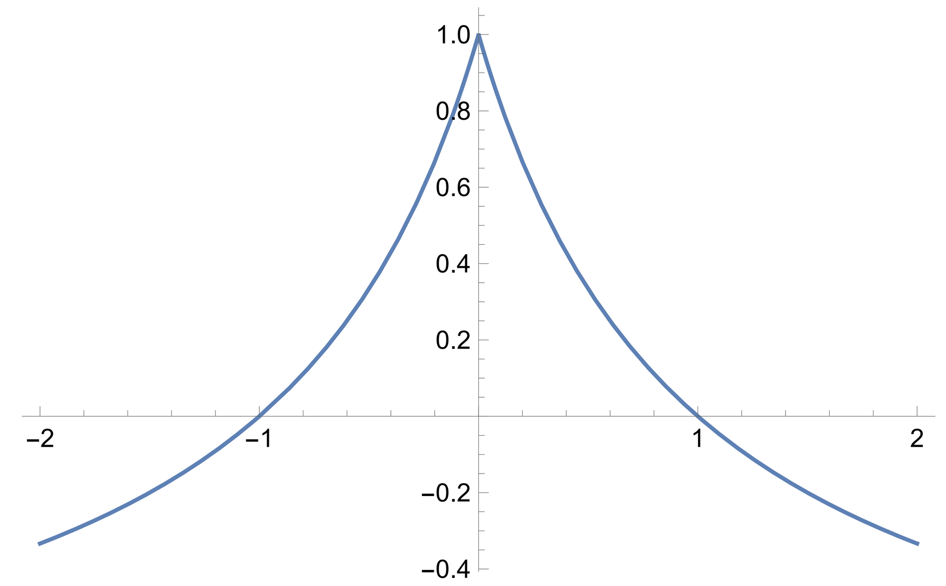


Figure 2: Piecewise function for correlations from same example from [1]. Derivative will be discontinuous at the critical point $g = g_c = 0$

QUANTUM PHASE TRANSITIONS

A QPT occurs at zero temperature, and describes a sudden change in the ground state of a system as a physical parameter, such as magnetic field strength or pressure, is varied.

Often, a nonanalyticity in the ground state energy is used to define a QPT [2]. However, Wolf et al. show by construction that **MPS are only guaranteed to satisfy an alternative definition of QPT**, namely the existence of discontinuities in correlations.

To find QPT in MPS we investigate the correlations

For MPS dependent on a continuous parameter, the relative size of eigenvalues will change with the parameter which will therefore affect the correlations.

An example graph of eigenvalues is shown in figure 1. When there is a crossover between different eigenvalue choices, a QPT occurs. Figure 2 shows that a discontinuity in the limit of the correlation function coincides with this crossover, identifying the QPT.

AN EXAMPLE

A simple example of the calculations described is given by the choice of MPS tensors:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & g \\ 0 & 0 \end{pmatrix}$$

where g is continuous and represents a changing physical parameter.

Here E_I has eigenvalues $\nu_+ = 1 + g$ and $\nu_- = 1 - g$, graphed in figure 1. An example correlation using σ_z, the observable associated with the spin along the z axis, is

$$\langle \sigma_i^z \sigma_{i+1}^z \rangle = \frac{\left(\frac{3(1+g)^2}{1-g}\right)(1-g)^N + (1-g)(1+g)^N}{3(1+g)(1-g)^N + (1+g)^{N+1}}$$

In the thermodynamic limit ($N \to \infty$) this gives us the functions:

$$\langle \sigma_i^z \sigma_{i+1}^z \rangle_{\nu_+} = \frac{1-g}{1+g} \qquad \langle \sigma_i^z \sigma_{i+1}^z \rangle_{\nu_-} = \frac{1+g}{1-g}$$

Graphing these functions over their defined domains gives figure 2, where the discontinuity in the derivative at the critical point is clear.

REFERENCES

[1] Michael M. Wolf, Gerardo Ortiz, Frank Verstraete, and J. Ignacio Cirac. Quantum phase transitions in matrix product systems. *Physical Review Letters*, 97(11), 2006.

[2] Subir Sachdev. *Quantum Phase Transitions*. Cambridge University Press, 1 2000.

[3] Nick G. Jones, Julian Bibo, Bernhard Jobst, Frank Pollmann, Adam Smith, and Ruben Verresen. Skeleton of matrix-product-state-solvable models connecting topological phases of matter. *Physical Review Research*, 3(3), 2021.

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