KZ functor for rational Cherednik algebras

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Motivation

Both rational Cherednik algebras and Iwahori-Hecke algebras are structures that appear extensively in the study of representation theory.

Wouldn't it be nice to have a way of linking the two?

Rational Cherednik algebra module

generated by x, y, t_w act on $\mathbb{C}[x]$-span$\{e_1, \dots, e_d\}$

↓ *KZ functor [Gin+03]*

Iwahori-Hecke algebra module

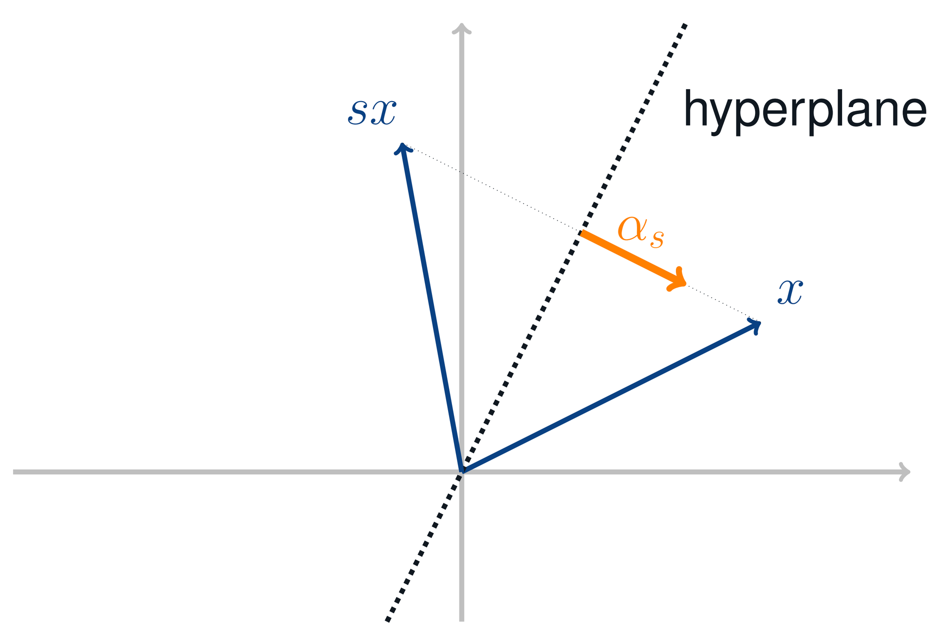
generated by T_s act on $\mathbb{C}$-span$\{e_1, \dots, e_d\}$

An answer is the Knizhnik-Zamolodchikov functor.

Complex reflection groups

Let V be a $\mathbb{C}$–vector space.

A linear transformation s is a *complex reflection* if it fixes a hyperplane $H \subseteq V$. A *complex reflection group* W is generated by complex reflections.



Pick a vector α_s orthogonal to H, which is also an eigenvector of s. From the dual vector space V^*, we then choose $\alpha_s^\vee : V \to \mathbb{C}$ such that $ker(\alpha_s^\vee) = H$.

After normalisation, our choice of α_s and $\alpha_s^\vee$ would then satisfy

$$sx = x - \alpha_s^\vee(x)\,\alpha_s \quad \forall x \in V.$$

Rational Cherednik algebras

Let W be a complex reflection group acting on a vector space V.
Let parameter $c_s \in \mathbb{C}$ for each reflection $s \in W$, such that $c_s = c_{wsw^{-1}}$ for all $w \in W$.
Consider the following two very similar algebras:

A *rational Cherednik algebra* $\tilde{\mathbb{H}}$ is generated by x, y, t_w for $x \in V, y \in V^*, w \in W$ with

$$t_w x = (wx)t_w, \quad t_w y = (wy)t_w$$
$$yx = xy + \langle x, y\rangle - \sum_{s \in R} c_s \langle x, \alpha_s^\vee\rangle\langle \alpha_s, y\rangle t_s. \quad \text{[Gri10]}$$

An algebra $\mathcal{D}(V) \rtimes W$ is generated by x, ∂_y, t_w for $x \in V, y \in V^*, w \in W$ with

$$t_w x = (wx)t_w, \quad t_w \partial_y = \partial_{(wy)} t_w$$
$$\partial_y x = x\partial_y + \langle x, y\rangle.$$

Proposition: $\tilde{\mathbb{H}}$ and $\mathcal{D}(V) \rtimes W$ are isomorphic as algebras via

$$y \mapsto \partial_y - \sum_{s \in R} c_s \langle \alpha_s, y\rangle \frac{1}{\alpha_s}(1 - t_s). \quad \text{[Gin+03, Theorem 5.6]}$$

Construct RCA-modules

For each representation (E, π) of W, where $E = span\{e_1, \dots, e_d\}$ and $\pi : W \to GL(E)$, a *RCA-module* can be induced with

$$t_w e_j = \pi(w) e_j$$
$$y e_j = 0 \qquad \text{for } j = 1, 2, \dots, d.$$

Each element in the module has the form

$$p_1(x_1, \dots, x_n)\, e_1 + \cdots + p_d(x_1, \dots, x_n)\, e_d\,.$$

Hecke algebras via monodromy

Let $a_0 \in V$ be a basepoint.
For $j = 1, 2, \dots, d$, let f_j be an element of a RCA-module such that

$$\partial_y f_j = 0 \quad \forall y \in V^* \qquad \textit{and} \qquad f_j(a_0) = e_j.$$

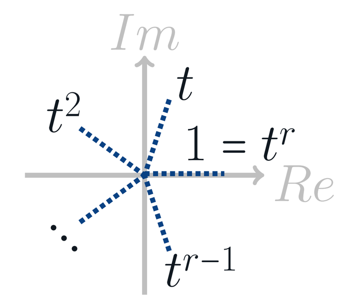
↓ partial differential equations ↓ initial conditions

For each reflection $s \in W$, define a monodromy matrix

$$T_s = \begin{pmatrix} | & & | \\ f_1(sa_0) & \cdots & f_d(sa_0) \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} & \pi(s) & \end{pmatrix}.$$

These matrices satisfy the Hecke relations of the Hecke algebra of W, and hence produces a Hecke module with generators T_s acting by the matrices above [Gin+03, Theorem 5.13].

An example: cyclic groups



Consider a cyclic group of order r

$$W = \{1, t, \dots, t^{r-1} \mid t^r = 1\}$$

acting on vector space $V = \mathbb{C}$.

The corresponding RCA $\tilde{\mathbb{H}}$ is generated by x_1, y_1, t with

$$x_1 = \zeta x_1 t, \quad t y_1 = \zeta^{-1} y_1 t$$
$$y_1 x_1 = x_1 y_1 + 1 - \sum_{\ell=1}^{r-1} c_\ell (1 - \zeta^\ell) t^\ell \qquad \text{where } \zeta = e^{2\pi i/r}.$$

From the RCA-module induced by the regular representation of W, we solve the PDEs to find

$$f_j = a_0^{-k_j} x_1^{k_j} e_j \qquad \text{where } k_i = \sum_{\ell=1}^{r-1} c_\ell (\zeta^{i\ell} - 1).$$

Then, the monodromy matrix for reflection $t \in W$ is

$$T_1 = \begin{bmatrix} \zeta^{-k_0} & & \\ & \ddots & \\ & & \zeta^{r-1-k_{r-1}} \end{bmatrix}.$$

This produces a Hecke module where the Hecke algebra generator T_1 acts by the matrix above and satisfies Hecke relation

$$\prod_{j=0}^{r-1} (T_1 - q_j) = 0 \qquad \text{with parameters } q_j = \zeta^{(j - k_j)}.$$

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References

[Gin+03] V. Ginzburg et al. "On the category $\mathcal{O}$ for rational Cherednik algebras". In: *Inventiones Mathematicae* 154.2 (2003), pp. 617–651.

[Gri10] S. Griffeth. "Towards a combinatorial representation theory for the rational Cherednik algebra of type $G(r, p, n)$". In: *Proceedings of the Edinburgh Mathematical Society* 53.2 (2010), pp. 419–445.